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# Application of Runge-Kutta Fourth Order (RK-4) Method to Solve Logistic Differential Equations

**Abstract**

In this paper, Runge-Kutta fourth order (RK-4) method is employed to obtain approximate solution of Logistic differential equations which are first order non-linear differential equations used in to model the growth of populations. The results show that method converges rapidly and approximates the exact solution very accurately.

**Keywords:** Runge-Kutta Fourth Order Method, C-language, Logistic Differential Equation, Stable and Unstable Problem.

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**Introduction**

Belgian Mathematician and Sociologist Pierre Francois Verhulst[1] was introduced Logistic differential equation to model the growth of populations limited by finite resources. The Logistic differential equation is given by

$$\frac{dP}{dt} = rP\left[1 - \frac{P}{K}\right] \ldots\ldots\ldots.(1)$$

Here $P(t)$ is called the population size at time $t$ and $\frac{dP}{dt}$ gives the change in population size over time $t$. (1) contains two positive parameters namely $r$ and $K$. The first parameter $r$ is called the growth parameter and second parameter is called the carrying capacity. Solow[2] used Logistic differential equation to discussed a contribution to the theory of economic growth.

Runge-Kutta fourth order (RK-4) method was developed around 1900 by the German mathematicians C. Runge and M.W. Kutta. The RK-4 method is a method of order four, meaning that the total accumulated error is on the order of $o(h^4)$ while the local truncation error is on the order of $o(h^5)$. A history of Runge-Kutta methods was given by Butcher[3]. Dormand and Prince[4] gave a family of embedded Runge-formulae. Zingg and Chisholm[5] discussed the Runge-Kutta method for linear ordinary differential equation. Milne[6] gave a note on the Runge-Kutta method. Cash and Karp[7] established a variable order Runge-Kutta method for initial value problems with rapidly varying right hand sides. Ralston[8] gave Runge-Kutta method with minimum error bounds. An order bound for Runge-Kutta method was given by Butcher[9]. Bogacki and Shampine[10] explained 3(2) pair of Runge-Kutta formulas. A modification of the Runge-Kutta fourth order method was given by Blum[11]. Cash[12] used a class of implicit Runge-Kutta methods for numerical integration of stiff ordinary differential equations. Mehdi and Kareem[13] solved L$\ddot{u}$ chaotic system using fourth order Runge-Kutta method. Yang and Sten [14] applied Runge-Kutta method for solving uncertain differential equations. Enright and Muir[15] used efficient classes of Runge-Kutta methods for two point boundary value problems. Application of the fourth order Runge-Kutta method for the solution of high-order general initial value problems was given by Cortell[16]. Yaakub and Evans[17] established a fourth order Runge-Kutta RK(4,4) method with error control. Estimating the error of the classic Runge-Kutta formula introduced by Hosea and Shampine[18]. A simplified derivation and analysis of fourth order Runge-Kutta method was given by Musa et.al.[19].

This paper uses Runge-Kutta fourth order(RK-4) method to solve Logistic differential equations. The advantage of this proposed method is its capability for obtaining exact solution without any difficulty and spending a very little time. The aim of this work is to establish exact solution or approximate solution of high degree of accuracy for Logistic differential equations using Runge-Kutta fourth order(RK-4) method.

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**Runge-Kutta Fourth Order (RK-4) method for First Order I.V.P.**

Consider the first order I.V.P. $\frac{dy}{dx} = f(x,y)$ …… (2) with $y(x_0) = y_0$ ……………(3)

By Runge-Kutta fourth order(RK-4) method, the sequence of approximation for $y$ is given by

$$y_{n+1} = y_n + \frac{1}{6}[k_1 + 2k_2 + 2k_3 + k_4]$$

$x_{n+1} = x_n + h$, for $n = 0,1,2,3,\ldots$

$$k_1 = hf(x_n, y_n)$$

$$k_2 = hf\left(x_n + \frac{h}{2}, y_n + \frac{k_1}{2}\right)$$

$$k_3 = hf\left(x_n + \frac{h}{2}, y_n + \frac{k_2}{2}\right)$$

$$k_1 = hf(x_n + h, y_n + k_3)$$

Here $h$ is the interval between equidistant values of $x$.

Logistic differential equation which is given by (1) with initial condition $P(t_0) = P_0$ can be treat as a first order initial value problem given by (2)&(3) and solved by the above discussed method.

**Stability and Conditioning**

If in an initial value problem, the small changes either in function $f$ or in the initial condition induces large effects on the solution of the problem then the problem is said to be ill-conditioned or unstable. Conversely, a problem is said to be well-conditioned or stable if small changes in the data induces small changes in the corresponding solution of problem.

A solution $y(x)$ of initial value problem (2) with initial condition (3) is said to be stable with respect to the initial condition (3) if, given any $\in > 0$, there is a $\delta > 0$ such that any other solution $\bar{y}(x)$ of (2) with initial condition (3) satisfying

$|y(x) - \bar{y}(x)| \leq \in$ whenever $|y(x_0) - \bar{y}(x_0)| \leq \delta$ for all $x > x_0$. ……………….(4)

**C-Program of Runge-Kutta Fourth Order (RK-4) Method for First Order Initial Value Problems**

```
#include<stdio.h>
#include<conio.h>
#include<math.h>
float f(float x, float y)
{return 0.08*y-0.00008*y*y+0*x;}
int main()
{ float x0,y0,h,k1,k2,k3,k4,y1,x;
int iter,i;
clrscr();
printf("Enter the value of x0 and y0 \n");
scanf("%f%f",&x0,&y0);
printf("Enter the value of h \n");
scanf("%f",&h);
printf("Enter the value of iteration");
scanf("%d",&iter) ;
for(i=1;i<=iter;i++)
{
x=x0+h;
 k1=h*f(x0,y0);
k2=h*f(x0+h*0.5,y0+k1*0.5);
k3=h*f(x0+h*0.5,y0+k2*0.5);
k4=h*f(x0+h,y0+k3);
y1=y0+0.16666*(k1+2*k2+2*k3+k4);
printf("the value of y=%f at x=%f\n",y1,x);
x0=x;
y0=y1;
}
getch();
return 0;
}
```

**Applications**

In this section, some applications are given in order to demonstrate the effectiveness of Runge-Kutta fourth order (RK-4) method to solve Logistic differential equations.

**Application: 1**

The Logistic differential equation(1) with growth parameter $r = 1$, carrying capacity $K = 10$ and $P(0) = 2$ is given by

$\frac{dP}{dt} = P\left[1 - \frac{P}{10}\right]$ ……………………(5)

with $P(0) = 2$………………………(6)

**Application: 2**

The Logistic differential equation(1) with growth parameter $r = 1$, carrying capacity $K = 1$ and $P(0) = 5$ is given by

$\frac{dP}{dt} = P[1 - \mathrm{P}]$ ……………………(7)

with $P(0) = 5$………………………(8)

**Application: 3**

The Logistic differential equation(1) with growth parameter $r = 0.08$, carrying capacity $K = 1000$ and $P(0) = 100$ is given by

$\frac{dP}{dt} = 0.08P\left[1 - \frac{P}{1000}\right]$ ……………………(9)

with $P(0) = 100$………………………(10)

**Application: 4**

The Logistic differential equation(1) with growth parameter $r = 0.25$, carrying capacity $K = 20$ and $P(0) = 1$ is given by

$\frac{dP}{dt} = 0.25P\left[1 - \frac{P}{20}\right]$ ……………………(11)

with $P(0) = 1$………………………(12)

**Output of Application: 1**

```
Enter the value of x0 and y0
0
2
Enter the value of h
0.1
Enter the value of iteration6
the value of y=2.164800 at x=0.100000
the value of y=2.339209 at x=0.200000
the value of y=2.523145 at x=0.300000
the value of y=2.716415 at x=0.400000
the value of y=2.918711 at x=0.500000
the value of y=3.129598 at x=0.600000
```

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**Output of Application: 2**

```
Enter the value of x0 and y0
0
5
Enter the value of h
0.1
Enter the value of iteration6
the value of y=3.621821 at x=0.100000
the value of y=2.898755 at x=0.200000
the value of y=2.455187 at x=0.300000
the value of y=2.156574 at x=0.400000
the value of y=1.942764 at x=0.500000
the value of y=1.782826 at x=0.600000
```

**Output of Application: 3**

```
Enter the value of x0 and y0
0
100
Enter the value of h
0.1
Enter the value of iteration6
the value of y=100.722282 at x=0.100000
the value of y=101.449188 at x=0.200000
the value of y=102.180748 at x=0.300000
the value of y=102.916977 at x=0.400000
the value of y=103.657898 at x=0.500000
the value of y=104.403534 at x=0.600000
```

**Output of Application: 4**

```
Enter the value of x0 and y0
0
1
Enter the value of h
0.1
Enter the value of iteration6
the value of y=1.024018 at x=0.100000
the value of y=1.048581 at x=0.200000
the value of y=1.073700 at x=0.300000
the value of y=1.099386 at x=0.400000
the value of y=1.125649 at x=0.500000
the value of y=1.152502 at x=0.600000
```

**Comparison between exact and RK-4 method solutions**
**Application: 1**

| $x$ | Exact Solution $y(x)$ | RK-4 Method Solution $y(x)$ |
|---|---|---|
| 0.1 | 2.164807 | 2.164800 |
| 0.2 | 2.339223 | 2.339209 |
| 0.3 | 2.523167 | 2.523145 |
| 0.4 | 2.716446 | 2.716415 |
| 0.5 | 2.918751 | 2.918711 |
| 0.6 | 3.129649 | 3.129598 |

**Application: 2**

| $x$ | Exact Solution $y(x)$ | RK-4 Method Solution $y(x)$ |
|---|---|---|
| 0.1 | 3.621482 | 3.621821 |
| 0.2 | 2.898421 | 2.898755 |
| 0.3 | 2.454919 | 2.455187 |
| 0.4 | 2.156362 | 2.156574 |
| 0.5 | 1.942594 | 1.942764 |
| 0.6 | 1.782688 | 1.782826 |

**Application: 3**

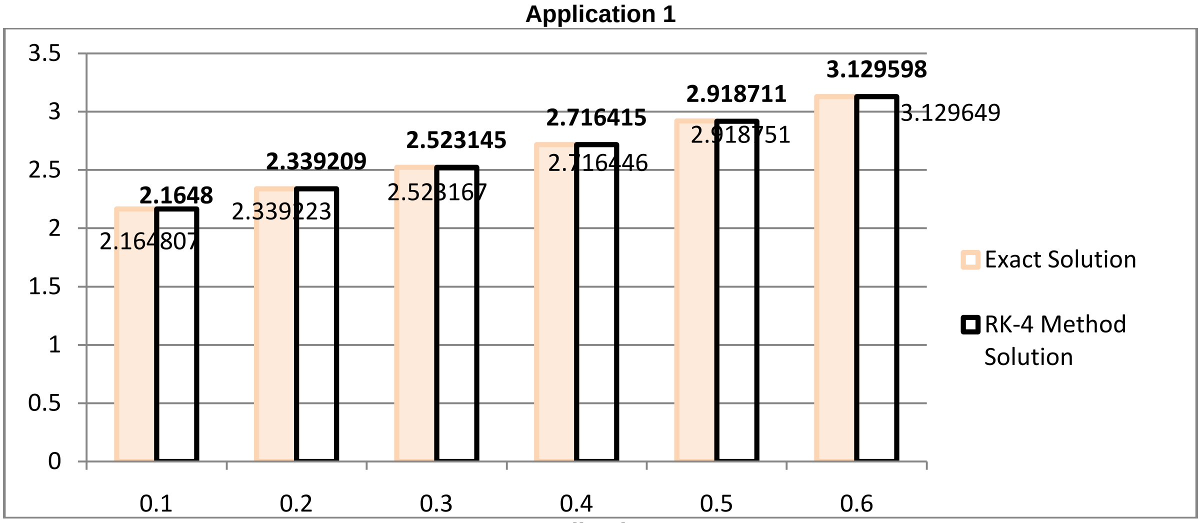
| $x$ | Exact Solution $y(x)$ | RK-4 Method Solution $y(x)$ |
|---|---|---|
| 0.1 | 100.722308 | 100.722282 |
| 0.2 | 101.449244 | 101.449188 |
| 0.3 | 102.180831 | 102.180748 |
| 0.4 | 102.917089 | 102.916977 |
| 0.5 | 103.658041 | 103.657898 |
| 0.6 | 104.403706 | 104.403534 |

**Application: 4**

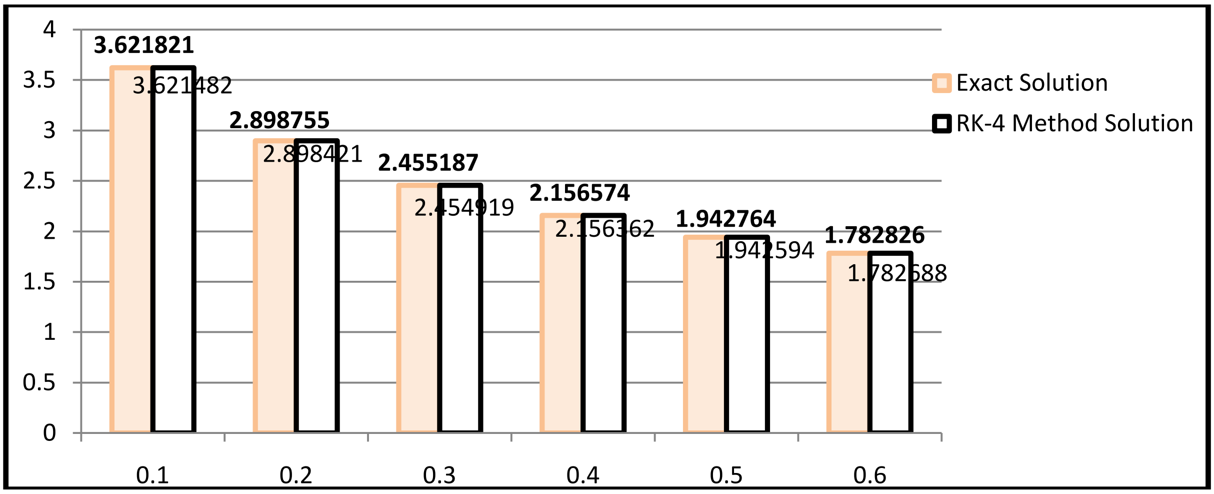
| $x$ | Exact Solution $y(x)$ | RK-4 Method Solution $y(x)$ |
|---|---|---|
| 0.1 | 1.024019 | 1.024018 |
| 0.2 | 1.048583 | 1.048581 |
| 0.3 | 1.073703 | 1.073700 |
| 0.4 | 1.099389 | 1.099386 |
| 0.5 | 1.125654 | 1.125649 |
| 0.6 | 1.152508 | 1.152502 |

**Comparison between Exact and RK-4 Method Solutions by Graphical Representation using above Data**

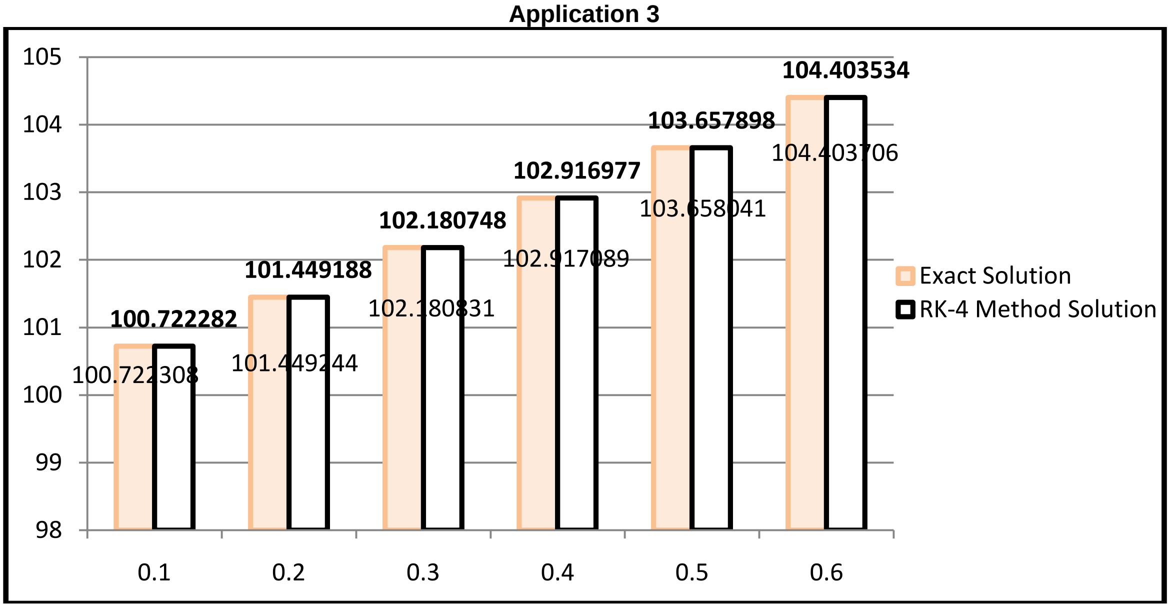
**Application 1**



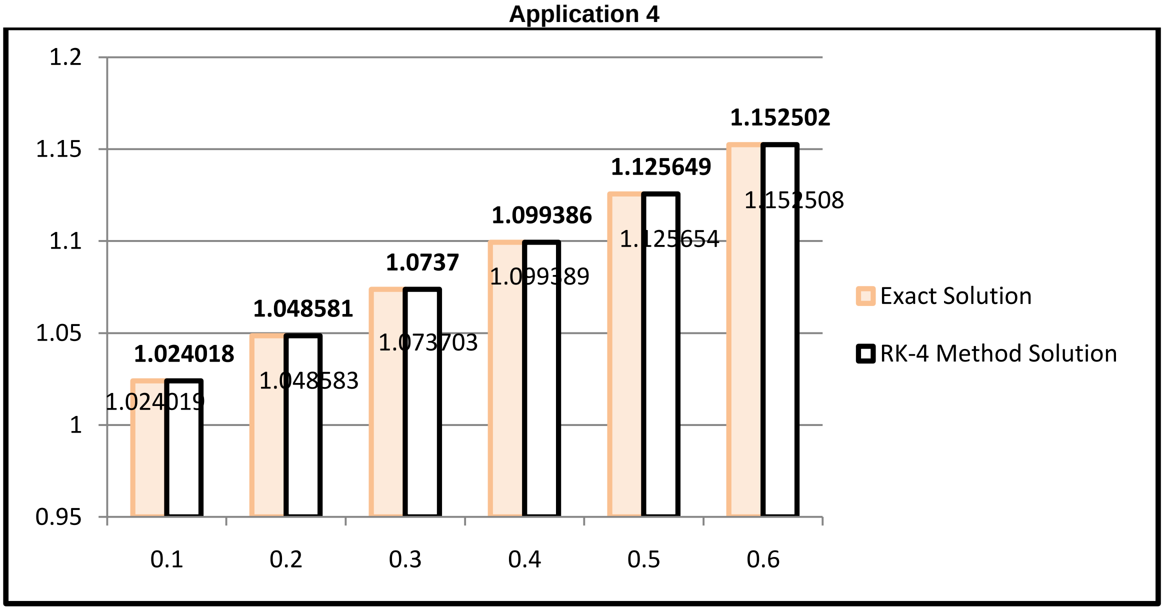

**Application 2**




Application 3




Application 4



**Conclusion**

In this paper, we have successfully developed the Runge-Kutta fourth order (RK-4) method to solve the Logistic differential equations and comparison between exact and RK-4 method solutions are given in graphical and tabular form. The given applications show that the RK-4 method needless computational work to obtained solution of Logistic differential equations with high degree of accuracy.

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